Noise-induced phase transitions in globally coupled active rotators

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We study the nonequilibrium phenomena in a globally coupled active rotator model with multiplicative noise. It is shown that at the critical values of the noise intensity the system undergoes noise-induced phase transitions and forms clusters. From the Fokker-Planck equation, which describes dynamics of the system, we analytically obtain the critical noise intensity at specific values of the parameter space. Numerical simulations reveal the phase transitions producing peculiar phase diagrams. The nature of the transitions is also discussed in detail.

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I. INTRODUCTION

From an intuitive point of view, random forces on a dynamical system give rise to disordering effects. In distinction, however, it has been discovered in recent years a certain class of nonequilibrium phenomena, the noise-induced phase transitions [1]. The external noise coupled to the state of the system may drastically change the stability of the system. In some cases, new stable states appear under the influence of strong external noise. One may find examples in a model of biology [2], chemical reactions [3], optics [4], plasma physics [5], etc., which have phenomenological justifications [6].

The macroscopic behavior of a globally coupled oscillator model has been studied in various systems. In the neuronal signal processing, the synchronous oscillations found in the visual cortex system have been modeled and understood via coupled phase oscillators [7]. One may approximate the Josephson-junction array as coupled phase oscillators to describe dynamics of the system [8]. Examples may also be found in chemical reaction systems [9] and charge-density waves [10].

The dynamics of coupled oscillators in the weak coupling limit has usually been investigated in the reduced model to the effective interaction given by the first Fourier mode [11]. It has been claimed, however [12,13], that higher Fourier mode interactions are indispensable for interesting collective dynamics. In [12,13] it has been shown that the phase oscillator system with interactions of higher harmonics eventually converges to the clustered states at some parameter range.

The question on the role of noise in the phase oscillator system has been raised continuously. It has been shown in [14] that the additive noise generates the switching phenomenon of the clustered states of the phase model with higher Fourier mode interactions. The additive noise has also been shown to affect the stability of the states [11,12]. In an active rotator model, a phase model of either a limit-cycle oscillator or an excitable element [15], with additive noise shows the transition from moving (excited) state to stationary (inhibited) state at a critical noise intensity.

In this paper we try to understand the dynamical effects of noise in the coupled active rotator model. Instead of introducing simple additive noise, we assume that the influence of the environment on the macroscopic properties of the system is described via the stochastic external parameter. We introduce a random fluctuation to the coupling strength and study the nonequilibrium phenomena that cannot be seen in the deterministic case or in the system with simple additive noise. We show that in the coupled active rotator system, only with the first Fourier mode does the multiplicative noise induces a bifurcation at a critical value of noise intensity, thus forming two stable clustered states. This is a pure noise effect and shows a route to the clustering phenomena without introducing higher Fourier mode interactions. The cases for the higher Fourier mode driving forces are also considered for completeness.

In the following section we describe the coupled active rotator model used in this paper. Section III is devoted to present the analytical study of the system without intrinsic frequency. Results of the numerical simulation are presented in Sec. IV, showing the phase diagrams of the system with the higher Fourier mode driving forces in addition to the first Fourier mode driving force. The nature of the phase transitions is also discussed with summarized results in Sec. V.

II. MODEL

A (noiseless) model of N coupled active rotators under study is expressed by the equation of motion [15,16]

$$\frac{d\phi_i}{dt} = \omega - b\sin(\nu\phi_i) - \sum_{j=1}^{N} K_{ij}\sin(\phi_i - \phi_j), \qquad (1)$$

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where ϕ_i, $i = 1, 2, \ldots, N$, is the phase of the ith rotator. ω is the intrinsic frequency that is uniformly given to each rotator. The second term on the right-hand side of Eq. (1) (from now on we denote this as the b term) is introduced to mimic the dynamics of stochastic limit-cycle oscillators or excitable elements [15,16]. We will consider the case for $\nu = 1$, 2, and 3 in this paper. The third term on the right-hand side of Eq. (1) describes global coupling, which depends on the phase difference of two rotators. In the absence of noise, if the coupling is ferromagnetic, i.e., $K_{ij} > 0$, then this term gives perfect synchrony, which means $\phi_i(t) = \phi(t)$ for all i. The steady state of Eq. (1) is the ground state of the system with the Hamiltonian

$$H = \sum_i \left(\omega\phi_i - \frac{b}{\nu}\cos(\nu\phi_i)\right) - \sum_{(ij)} K_{ij}\cos(\phi_i - \phi_j), \tag{2}$$

where the (ij) summation is taken over all pairs (each pair counted only once). For the case of $\nu = 1$ and ferromagnetic coupling, ϕ_i dwells on two phases. When $|b/\omega| > 1$, Eq. (2) has a local minimum at $\phi_i = \phi_0 \equiv \sin^{-1}(w/b)$ and each element is at the stable fixed point. When $|b/\omega| < 1$, Eq. (2) has no local minimum, implying that the system is on the moving phase, i.e., each ϕ_i is a rotator with frequency $\sqrt{\omega^2 - b^2}$. The b term of Eq. (1) characterizes the system whether it is on the stationary state or on the moving state.

Now we assume the uniform ferromagnetic interaction $K_{ij} = K/N > 0$. If the system is coupled to a fluctuating environment, the coupling strength then may be assumed to be a stochastic quantity, which implies

$$\frac{K}{N} \to \frac{1}{N}[K + \sigma\eta_i(t)],$$

where $\eta_i(t)$ is a Gaussian white noise characterized by

$$\langle \eta_i(t)\rangle = 0,$$

$$\langle \eta_i(t)\eta_j(t')\rangle = 2\delta_{ij}\delta(t - t')$$

and σ measures the intensity of the noise. Thus Eq. (1) is replaced by the stochastic differential equation

$$\frac{d\phi_i}{dt} = \omega - b\sin(\nu\phi_i) - \frac{1}{N}[K + \sigma\eta_i(t)]\sum_{j=1}^{N}\sin(\phi_i - \phi_j). \tag{3}$$

Throughout this paper we set $K = 1$ using a suitable time unit.

III. ANALYTICAL STUDY OF THE FOKKER-PLANCK EQUATION

The macroscopic behavior of the system can be described by the probability distribution $P(\phi, t)$ of ϕ_i at time t, whose evolution is governed by the Fokker-Planck equation [17]. In the large-N limit, the Ito interpretation of the stochastic differential equation (3) yields the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial\phi}\left[\left\{\omega - b\sin(\nu\phi) - \int d\phi'\sin(\phi - \phi')n(\phi', t) + \sigma^2\int d\phi'\sin(\phi - \phi')n(\phi', t) \times \int d\phi''\cos(\phi - \phi'')n(\phi'', t)\right\}P(\phi, t)\right] + \sigma^2\frac{\partial^2}{\partial\phi^2}\left[\left\{\int d\phi'\sin(\phi - \phi')n(\phi', t)\right\}^2 P(\phi, t)\right], \tag{4}$$

where $n(\phi, t)$, the normalized number density of the rotators with phase ϕ at time t, is given by

$$n(\phi, t) = \frac{1}{N}\sum_{i=1}^{N}\delta(\phi_i(t) - \phi).$$

Since ϕ_i's are statistically independent for the uniform interaction $P(\phi, t)$ may be identified with $n(\phi, t)$. In this paper we will analyze the steady state of $n(\phi, t)$.

When $\omega = 0$, the steady state of Eq. (4) is given by the stationary solution

$$n(\phi) = \exp[-U_\nu(\phi - \alpha)]\left\{n(0)\exp[U_\nu(\alpha)] + \frac{J}{\sigma^2(C_n^2 + S_n^2)}\int_{-\alpha}^{\phi-\alpha} d\phi'\frac{\exp[U_\nu(\phi' - \alpha)]}{\sin^2\phi'}\right\}, \tag{5}$$

where

$$U_\nu(\phi) = \frac{1}{\sigma^2\sqrt{C_n^2 + S_n^2}}\ln\left|\frac{1 - \cos\phi}{\sin\phi}\right| + \ln|\sin\phi| + \frac{b}{\sigma^2(C_n^2 + S_n^2)}\int_0^{\phi} d\phi'\frac{\sin\nu(\phi' + \alpha)}{\sin^2\phi'} \tag{6}$$

and J is the constant probability current, which is determined by the boundary condition

$$n(\phi + 2\pi) = n(\phi), \tag{7}$$

with $\alpha \equiv \tan^{-1}(S_n/C_n)$. Here C_n and S_n are obtained by the self-consistent equations

$$C_n = \int_0^{2\pi} d\phi\cos\phi\; n(\phi),$$

$$S_n = \int_0^{2\pi} d\phi\sin\phi\; n(\phi). \tag{8}$$

In the case of $\nu = 1$ the boundary condition (7) gives

$J = 0$ because the integral of the last term in Eq. (6) is periodic, leading to $U_\nu(\phi + 2\pi) = U_\nu(\phi)$. When $J = 0$ the self-consistent equations (8) reduce to

$$\langle \cos\phi \rangle = \sqrt{C_n^2 + S_n^2}, \quad \langle \sin\phi \rangle = 0, \tag{9}$$

where $\langle\,\rangle$ means an average over ϕ with the weight function $\exp[-U_\nu(\phi)]$. In this case we can separate $U_1(\phi)$ into symmetric and antisymmetric parts as $U_1(\phi) = U_1^s(\phi) + U_1^a(\phi)$ with

$$U_1^s(\phi) = \frac{C_n^2 + S_n^2 + bC_n}{\sigma^2(C_n^2+S_n^2)^{3/2}} \ln\left|\frac{1-\cos\phi}{\sin\phi}\right| + \ln|\sin\phi|,$$

$$U_1^a(\phi) = -\frac{bS_n}{\sigma^2(C_n^2+S_n^2)^{3/2}}\frac{1}{\sin\phi}.$$

Then we obtain

$$\langle \sin\phi \rangle = Z^{-1}\int_0^\pi \exp[-U_1(\phi)]\sin\phi\{1-\exp[2U_1^a(\phi)]\}d\phi,$$

with $Z \equiv \int_0^{2\pi}\exp[-U_1(\phi)]d\phi$ leading to

$$\mathrm{sgn}(\langle\sin\phi\rangle) = \mathrm{sgn}(S_n).$$

Thus the self-consistent equations (9) give $S_n = 0$ and thus

$$U_1(\phi) = \frac{C_n + b}{\sigma^2 C_n^2}\ln|1-\cos\phi| + \frac{\sigma^2C_n^2 - C_n - b}{\sigma^2C_n^2}\ln|\sin\phi|. \tag{10}$$

$U_1(\phi) \to -\infty$ as $\phi \to 0$ or 2π. When $\sigma^2C_n^2 > b + C_n$, $U_1(\phi) \to -\infty$ as $\phi \to \pi$. (See Fig. 1.) When $\sigma^2C_n^2 = b + C_n$, self-consistent equations (9) give $C_n = 1$, implying a continuous transition from a one-cluster state to a two-cluster state: While for $\sigma < \sigma_c \equiv \sqrt{1+b}$ $n(\phi)$ has a peak leading to a one-cluster state, for $\sigma > \sigma_c$ it has two peaks leading to a two-cluster state.

Since the Fokker-Planck equation (4) is invariant under the transformation of ϕ into $-\phi$, we can assume $n(\phi)$ as an even function, i.e., $n(-\phi) = n(\phi)$. This symmetry leads to the conditions $J = S_n = 0$ and thus

$$U_2(\phi) = \frac{1}{\sigma^2C_n}\ln|1-\cos\phi| + \frac{\sigma^2C_n^2 + 2b - C_n}{\sigma^2C_n^2}\ln|\sin\phi|,$$

$$U_3(\phi) = \frac{C_n + 3b}{\sigma^2C_n^2}\ln|1-\cos\phi| + \frac{\sigma^2C_n^2 - C_n - 3b}{\sigma^2C_n^2}\ln|\sin\phi|. \tag{11}$$

The above equations lead to the transitions at $\sigma_c = \sqrt{1-2b}$ and $\sqrt{1+3b}$ for $\nu = 2$ and 3, respectively, because $C_n = 1$ at the transition points: While for $\sigma < \sigma_c$ $n(\phi)$ has a cluster, for $\sigma > \sigma_c$ it has two clusters. Figure 2 shows the phase diagram for $\nu = 1$, 2, and 3. While the phase boundaries of $\nu = 1$ and 3 are similar, the phase boundary of $\nu = 2$ is different from the others: As b increases, σ_c also increases for $\nu = 1$ and 3, but for $\nu = 2$, σ_c decreases. This different trend comes from the difference of the role of the b term. For $\nu = 2$ the b term prefers the two-cluster state because the b term is symmetric under $\phi \to \phi + \pi$. However, for $\nu = 1$ and 3 the b term prefers the one-cluster state.

IV. NUMERICAL RESULTS

To investigate phase transitions at finite ω, we have performed a numerical simulation of Eq. (3). In the simulation, we have used the second-order Runge-Kutta method with discrete time steps of $\Delta t = 0.01$ with random initial configurations. At each run, the first 4×10^4 time steps per spin have been discarded to achieve steady state and 10^5 time steps per spin have been used to compute averages. We have restricted ω as a unit and considered the system of size $N = 1000$.

First, we consider the case $\nu = 1$. Figure 3 shows the phase diagram with $\omega = 1$. When $\sigma = 0$, the system has a transition point at $b = 1$. For $b > 1$, the system is on a steady state fixed point. For $b < 1$, the rotators are all synchronized so that they move in a uniform phase velocity. As σ increases, the phase structure persists up to some critical value of σ, σ_c, which depends on b. The region denoted by S_s (S_d) is where all the elements of the system are at one (two) fixed angle(s). In the P_s (P_d) state the elements are in a uniform periodic motion

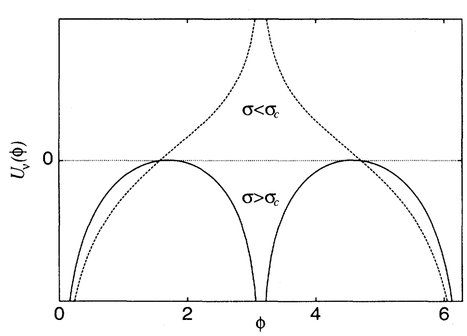

FIG. 1. Schematic diagram of U_ν. The solid line is for $\sigma > \sigma_c$ and the dashed line is for $\sigma < \sigma_c$.

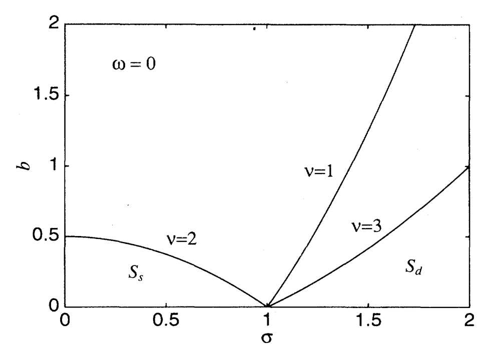

FIG. 2. Phase diagram for $\omega = 0$. S_s and S_d represent one-cluster and two-cluster phases, respectively.

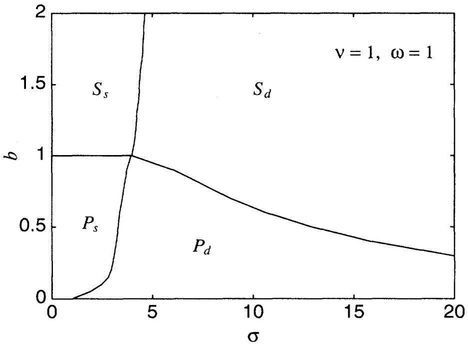

FIG. 3. Phase diagram obtained by numerical simulation performed for the system of size $N = 1000$ when $\omega = 1$ and $\nu = 1$. S_s (P_s) and S_d (P_d) represent one-cluster stationary (moving) and two-cluster stationary (moving) phases, respectively.

grouping into one (two) cluster(s). On the $\sigma = \sigma_c$ line a phase transition occurs. The single steady fixed point bifurcates to two steady fixed points so that the system is split into two clusters, which dwell on either fixed point. The periodic motion of the system also bifurcates at the critical value of σ. The wholly synchronized rotators are split into two synchronized clusters.

In the region denoted by S_d in Fig. 3, there exist two stable clusters of rotators. The location of two stationary states differs by π. One may make an analogy to the spin system to understand this phenomena. The Gaussian white noise with mean zero in the coupling yields the antiferromagnetic interactions as well as the ferromagnetic ones. The system therefore has an approximate symmetry $\phi_i \to \phi_i + \pi$ when σ is large. When this symmetry is exact, i.e., in the $\sigma \to \infty$ limit, one expects an equal intensity of the two clusters. The same argument holds for the periodic regions. In Fig. 4(d) one can observe the propagating secondary peak, which is at a distance π from the primary one. As σ increases, the intensities of the two peaks become equal. Figures 4(a) and 4(c) show the time evolution of steady state of $n(\phi, t)$ in the regions of S_s and P_s, respectively.

As σ gets larger, the transition $S \to P$ occurs at smaller value of b, as can be seen in Fig. 3. This reflects the fact that the random force diminishes the pinning effect generated by the b term so that the stationary state can make a transition to the moving state at a smaller value of b. In contrast to the case of simple additive noise case, which has been studied by Shinomoto and Kuramoto [15], there is no nonanalyticity in the phase boundary. Rather, it continues to the infinite value of σ. When $b < 1$, one can observe the transitions $P_s \to P_d$ and $P_d \to S_d$ consecutively as σ increases.

The phase diagram is presented in Fig. 5 when $\nu = 2$. The phase transition of the one-cluster (stationary, moving) state to the two-cluster state (stationary, moving) occurs at a smaller value of noise intensity than in the case of $\nu = 1$. Contrary to the case when $\nu = 1$, the b term in Eq. (3) may generate an antiferromagnetic effect.

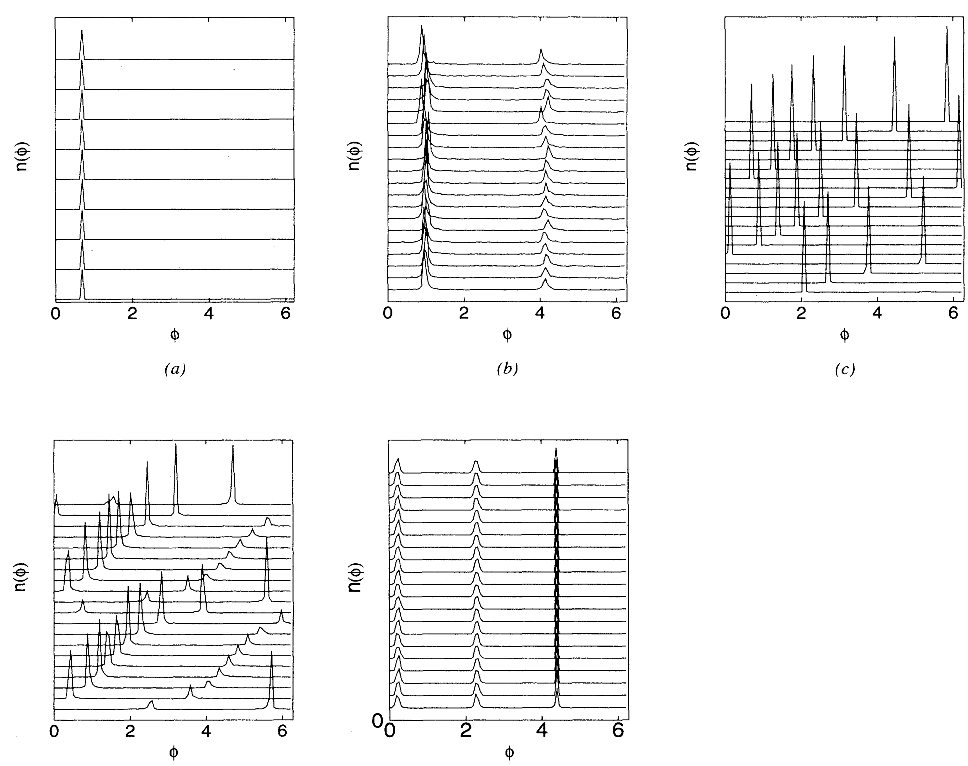

FIG. 4. Time evolutions of $n(\phi, t)$ at various phases in steady state: at (a) $\nu = \omega = 1$, $b = 1.5$, and $\sigma = 2$ in the one-cluster stationary phase (S_s); (b) $\nu = \omega = 1$, $b = 0.5$, and $\sigma = 15$ in the two-cluster stationary phase (S_d); (c) $\nu = \omega = 1$, $b = 0.5$, and $\sigma = 2$ in the one-cluster moving phase (P_s). (d) $\nu = \omega = 1$, $b = 0.5$, and $\sigma = 9$ in the two-cluster moving phase (P_d); (e) $\nu = 3$, $\omega = 1$, $b = 1.5$, and $\sigma = 2$ in the three-cluster stationary phase (S_t).

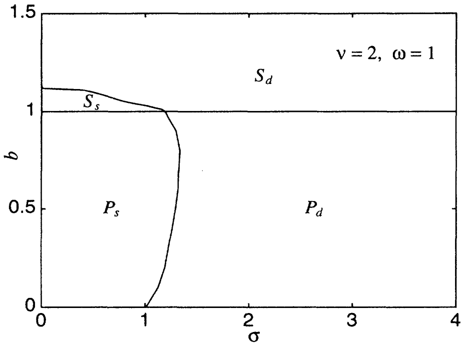

FIG. 5. Phase diagram obtained by numerical simulation performed for the system of size $N = 1000$ when $\omega = 1$ and $\nu = 2$. S_s (P_s) and S_d (P_d) represent one-cluster stationary (moving) and two-cluster stationary (moving) phases, respectively.

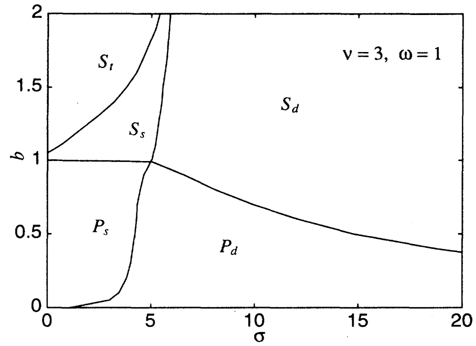

FIG. 6. Phase diagram obtained by numerical simulation performed for the system of size $N = 1000$ when $\omega = 1$ and $\nu = 3$. S_s (P_s) and S_d (P_d) represent one-cluster stationary (moving) and two-cluster stationary (moving) phases, respectively. S_t also means three-cluster stationary phase.

Therefore, the phase transition becomes more sensitive to the noise term than the case of $\nu = 1$.

One can see that when $\sigma = 0$ there are three stable states according to the value of b. The single stationary state in the $\nu = 1$ case is now split into single stationary and double stationary states. This is also due to the antiferromagnetic effects of the b term in Eq. (3). In fact, the S_d state is not a ground state of Eq. (2) but a metastable state. However, this metastable state has a large basin of attraction and thus the system initiated from the random configuration goes to the metastable state in steady state. Also, the multiplicative noise considered in this paper makes the metastability stronger and thus the state becomes a globally stable state.

One notices that in Fig. 5 the critical transition line from the stationary states to the moving states is almost independent of σ, i.e., characterized by the $b = 1$ line. This comes from the fact that the b term is symmetric under the translation of ϕ by π. When $\nu = 2$, therefore, the noise term does not play a significant role in the transition from stationary states to moving states as in the $\nu = 1$ case.

The results of $\nu = 3$ case are depicted in Fig. 6, which is similar to the case of $\nu = 1$. When $\sigma = 0$, the system has three stationary states. S_t stands for the stationary state with three clusters [see Fig. 4(e)]. When the b term dominates the interaction, the system is on the state of three stationary clusters. This state is also a metastable state with a larger basin of attraction. In contrast to the case of $\nu = 2$, the multiplicative noise reduces the basin of attraction, eventually leading to the transition from a three-cluster state to a one-cluster state. The transition from a one-cluster state to a two cluster state $S_s \rightarrow S_d$ and $P_s \rightarrow P_d$ is also observed. In Fig. 6, P_t, the moving state with three clusters, does not appear. P_t is expected when $b < 1$, since it is a moving state. At the same time, the b term should dominate the interaction to make more than one cluster. For $\omega = K = 1$, which is considered in this paper, there is no overlapping range of b and σ that satisfies both conditions. It may be possible, however, to obtain the P_t state in another range of parameters. The b term yields a more ferromagnetic effect than an antiferromagnetic one for $\nu = 3$. Therefore the critical line from the stationary states to the moving states sensitively depends on σ as for the $\nu = 1$ case. On the basis of these results, we expect that the results for any harmonics with b odd (even) resemble the ones for $\nu = 1$ (2).

V. DISCUSSION

In this paper we considered the nonequilibrium phenomena in a coupled active rotator model with multiplicative noise. The stable steady states of the number density of the rotators were analyzed both analytically and numerically. The key result in this paper is that at the critical noise intensity the system is split into two clusters without higher Fourier mode coupling, which is a pure noise effect. Therefore, our results exhibit another route to the clustering phenomenon, which cannot be seen in the deterministic or simple additive noise case.

We also considered the model with higher Fourier modes of an external source term. One may anticipate a more complicated rich structure adding higher Fourier modes of interaction terms. It would be interesting if our results can be tested in physiological systems. It will also be interesting to investigate the dynamics of clusters. This may be done by introducing some realistic parameters such as the time delay of interactions [18].

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